Network inference for truncated gaussian data

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Introduction

- Infer network defined by conditional dependencies

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- Infer network defined by conditional dependencies

- Gaussian variables zero inflated by double truncation
- Applications for the modelling of interactions between microbacterial populations
  - left truncation: replication phenomena
  - right truncation: non restrictive hypothesis for theoretical results !

Content

1. Modelling
2. Estimation process
   - Covariance matrix estimation
   - Precision matrix estimation
3. Theoritical results
4. Simulation studies

Modelling

Modelling

Let $\mathbf{X} \sim \mathcal{N}_p(0, \Sigma^*)$ be a gaussian p-vector with $\Sigma^*_{jj} = 1$ for all j. $\mathbf{X}$ is not observed but we observe $\mathbf{Y}$ defined from $\mathbf{X}$ by:
$Y_j = \mathbb{1}_{a_j \leq X_j \leq b_j} X_j$ where $a_j < b_j$.

A non directed graph structure is given by the inverse of the covariance matrix $\Theta^* := (\Sigma^*)^{-1}$:

$$\begin{aligned} X_j \longleftrightarrow X_k &\iff X_j \not\perp\!\!\!\perp X_k \mid (X_l)_{l \neq j,k} \\ &\iff \mathrm{cor}(X_j, X_k \mid (X_l)_{l \neq j,k}) \neq 0 \\ &\iff \Theta^*_{jk} \neq 0. \end{aligned}$$

Aim

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Infer the latent graph structure given by the precision matrix Θ^* of the gaussien vector $\mathbf{X}$ from the observations of the truncated vector $\mathbf{Y}$.

Theorotical tools - Pairs likelihood

Let us look at the likelihood $\mathcal{L}_{jk}(\Sigma^*_{jk}, y)$ of the pairs $(Y_j, Y_k),\ j < k$ where :

$$\mathcal{L}_{jk}(\sigma, y) = \sum_{a,b=0}^{1} \phi_{ab,jk}(\sigma, y_j, y_k) n_{ab}(y_j, y_k),$$

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- $\phi_{01,jk}(\sigma, y_j, y_k) = \phi_{01,jk}(\sigma, y_k) = \int_{[a_j, b_j]^c} f(x, y_k, \sigma)\mathrm{d}x$
- $\phi_{10,jk}(\sigma, y_j, y_k) = \phi_{10,jk}(\sigma, y_j) = \int_{[a_k, b_k]^c} f(y_j, y, \sigma)\mathrm{d}y$

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- $\phi_{00,jk}(\sigma, y_j, y_k) = \phi_{00,jk}(\sigma) = \iint_{[a_j, b_j]^c \times [a_k, b_k]^c} f(x, y, \sigma) \mathrm{d}x\mathrm{d}y.$

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Context

Let $\mathbf{Y} := (Y^{(1)}, \ldots, Y^{(n)})$ be an i.i.d. n-sample du vecteur Y. For every $j < k$, let us denote:

- $\mathcal{L}_{jk}^{(n)}(\Sigma_{jk}^*, \mathbf{y})$ the likelihood of the pairs sample $((Y_j^{(i)}, Y_k^{(i)}))_{1 \leq i \leq n}$ given by:

$$\mathcal{L}_{jk}^{(n)}(\sigma, \mathbf{y}) = \prod_{i=1}^{n} \mathcal{L}_{jk}(\sigma, y^{(i)}).$$

- $L_{jk}^{(n)}(\Sigma_{jk}^*, \mathbf{y})$ the log-likelihood $L_{jk}^{(n)}(\Sigma_{jk}^*, \mathbf{y}) = \log(\mathcal{L}_{jk}^{(n)}(\sigma, \mathbf{y}))$

Step 1: covariance matrix Σ^*

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Estimator of the covariance matrix

Estimator $\widetilde{\Sigma}^{(n)}$ of Σ^* is defined by:

$$\widetilde{\Sigma}_{jk}^{(n)} = \underset{|\sigma|\leq 1}{\operatorname{argmax}}\ L_{jk}^{(n)}(\sigma, \mathbf{y}). \tag{1}$$

Step 2: precision matrix Θ^*

Idea : estimate Θ^* thanks to the graphical Lasso (Friedman et al. (2007)) for the GGM

$\rightsquigarrow$ is to maximize the penalized log-likelihood of the Gaussian model.

$$\hat{\Theta}^{(n)} = \underset{\Theta \succ 0}{\operatorname{argmax}} \ \log\det(\Theta) - \operatorname{trace}(\Theta\hat{\Sigma}^{(n)}) - \lambda_n ||\Theta||_{1,\text{off}}. \tag{2}$$

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Tools

Step 1: Mei & al (2017), *under suitable assumptions, good properties of the population risk can be carried to the empirical risk, even in an non-convex case*
$\rightsquigarrow$ *control of* $|\widetilde{\Sigma}^{(n)}_{j,k} - \Sigma^*_{j,k}|$

Tools

Step 1: Mei & al (2017),
$\rightsquigarrow$ control of $|\widetilde{\Sigma}^{(n)}_{j,k} - \Sigma^*_{j,k}|$

Step 2: Ravikumar & al (2011), *estimating the concentration matrix under sparsity conditions without specific distributional assumptions, but rather analyze the estimator in terms of the tail behavior of* $\max_{j,k} |\widetilde{\Sigma}^{(n)}_{j,k} - \Sigma^*_{j,k}|$

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(H2) Let $j < k$. Let g be a function:
$\sigma \in [-1+\delta, 1-\delta] \mapsto \mathbb{E}\Big(L^{(n)}_{jk}(\sigma, \mathbf{y})\Big)$.

- $(-1+\delta)$ et $(1-\delta)$ are not critical points of g,
- g admits a finite number of critical points,
- all the critical points of g, different from Σ^*_{jk}, are non-degenerated, that is:

$$\sigma \neq \Sigma^*_{jk}, \quad g'(\sigma) = 0 \Rightarrow g''(\sigma) \neq 0.$$

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(H3) Technical (not written). The underlying intuition is to limit the influence of "non-edged" terms on edge terms.

Intermediate result on the estimated covariance matrix $\widetilde{\Sigma}^{(n)}$

Proposition

Under **(H1)** and, **(H2)**, let $0 < \rho < 1$. There exist known constants B, C and D such that if n satisfies $\frac{n}{\log n} \geq C \log\left(\frac{B}{\rho}\right)$, then the estimated covariance matrix $\widetilde{\Sigma}^{(n)}$ satisfies:

$$\mathbb{P}\left(||\widetilde{\Sigma}^{(n)} - \Sigma^*||_\infty \geq D\sqrt{\frac{\log n}{n} \log\left(\frac{B}{\rho}\right)} \right) \leq \frac{p(p-1)}{2}\rho,$$

where $||A||_\infty = \max_{j,k \in \{1,\ldots,p\}} |A_{jk}|$ is the infinite norm of matrix $A \in \mathbb{R}^{p^2}$.

$\rightsquigarrow$ Mei *et al.* (2017)

Final result about $\widetilde{\Theta}^{(n)}$

Theorem

Under **(H1)**, **(H2)** et **(H3)**. Let $c > 2$. There exist known constants B, C and D such that for all n satisfying $\frac{n}{\log n} > f(B, C, D, c, \Sigma^*)$ and $\lambda_n \propto \sqrt{\frac{\log n}{n} \log\left(Bp^c\right)}$, we have with probability $1 - \frac{1}{p^{c-2}}$:

(a) $||\widetilde{\Theta}^{(n)} - \Theta^*||_\infty \leq D\sqrt{\frac{\log n}{n} \log\left(Bp^c\right)}$.

(b) $E(\widetilde{\Theta}^{(n)}) \subset E(\Theta^*)$ and vertices (j,k) is correctly detected as soon as: $\left|\Theta^*_{jk}\right| > D\sqrt{\frac{\log n}{n} \log\left(Bp^c\right)}$.

$\rightsquigarrow$ Ravikumar *et al.* (2011)

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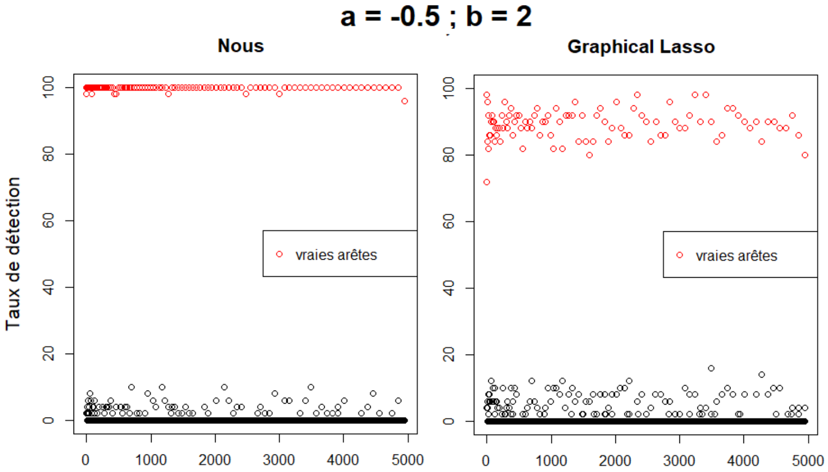
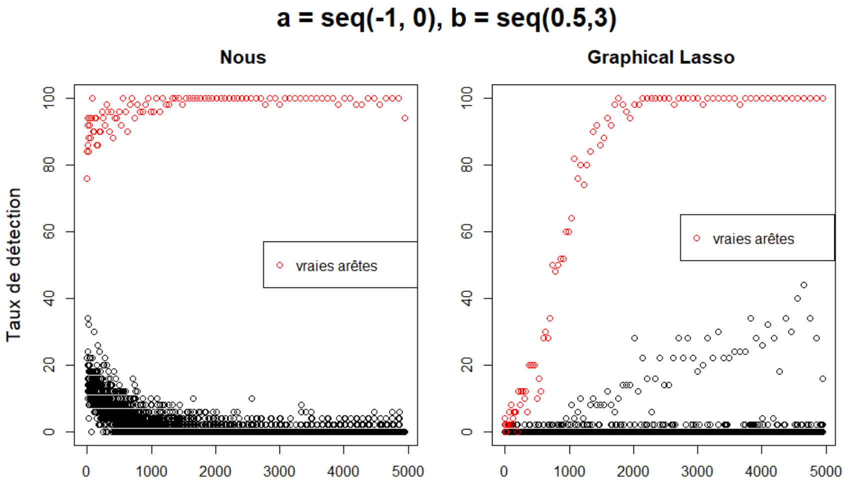
  - ➢ Identical thresholds : $a = -0.5, b = 2$
  - ➢ Non identical thresholds $a = seq(-1, 0), b = seq(0.5, 3)$
- Detection rate over 50 repetitions and comparaison with naive graphical Lasso directly on truncated data $\mathbf{Y}$.


a = -0.5 ; b = 2
Nous
Graphical Lasso
Taux de détection
vraies arêtes
vraies arêtes
0
20
40
60
80
100
0
1000
2000
3000
4000
5000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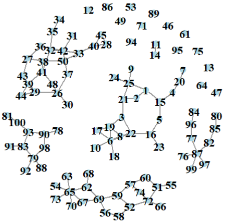
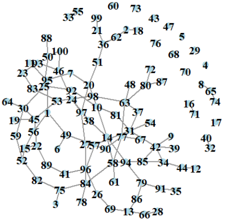
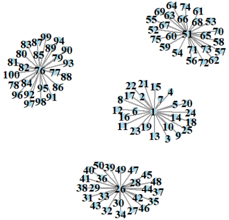


(a) Structure “cluster”.



(b) Structure “random”.



(c) Structure “hub”.

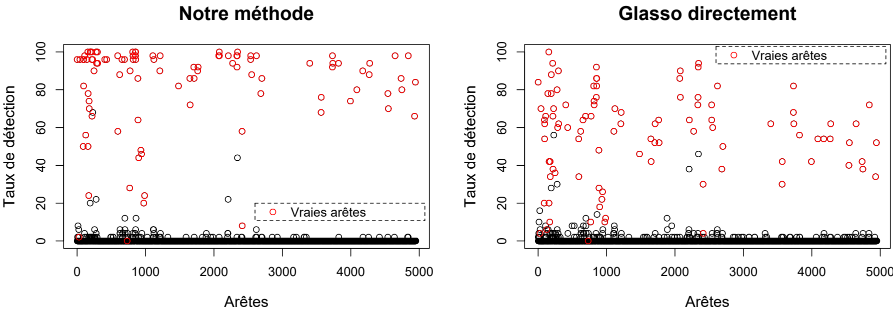
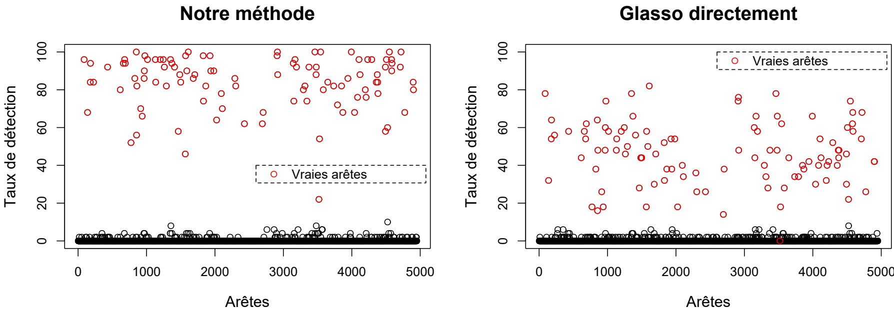


Figure: Detection rates obtained by our method (estimated truncation points) and by the graphical Lasso applied directly to the truncated data. The configurations of the truncation points are "identical only". Detection rates are obtained over 50 independent repetitions for $n = 500$ observations of $p = 100$ variables. "cluster" structure; red = true edges.



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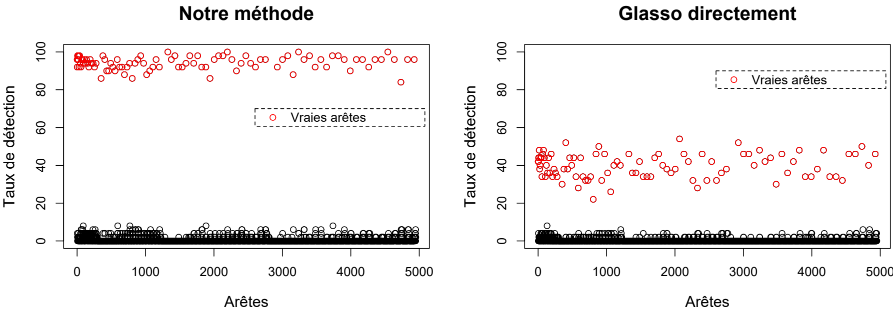

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Grazie mille !

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